

L^p -Spectral theory of locally symmetric spaces with \mathbb{Q} -rank one

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Abstract

We study the L^p -spectrum of the Laplace-Beltrami operator on certain complete locally symmetric spaces $M = \Gamma \backslash X$ with finite volume and arithmetic fundamental group Γ whose universal covering X is a symmetric space of non-compact type. We also show, how the obtained results for locally symmetric spaces can be generalized to manifolds with cusps of rank one.

Keywords: Arithmetic lattices, heat semigroup on L^p -spaces, Laplace-Beltrami operator, locally symmetric space, L^p -spectrum, manifolds with cusps of rank one.

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1 Introduction

Our main concern in this paper is to study the L^p -spectrum $\sigma(\Delta_{M,p}), p \in (1, \infty)$, of the Laplace-Beltrami operator on a complete non-compact locally symmetric space $M = \Gamma \backslash X$ with finite volume, such that

- (i) X is a symmetric space of non-compact type,
- (ii) $\Gamma \subset \text{Isom}^0(X)$ is a torsion-free arithmetic subgroup with $\mathbb{Q}\text{-rank}(\Gamma) = 1$.

We also treat the case of manifolds with cusps of rank one which are more general than the locally symmetric spaces defined above.

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Whether the L^p -spectrum of a complete Riemannian manifold M depends on p or not is related to the geometry of M . More precisely, Sturm proved in [23] that the L^p -spectrum is p -independent if the Ricci curvature of M is bounded from below and the volume of balls in M grows uniformly subexponentially (with respect to their radius). This is for example true if M is compact or if M is the n -dimensional euclidean space \mathbb{R}^n .

On the other hand, if the Ricci curvature of M is bounded from below and the volume density of M grows exponentially in every direction (with respect to geodesic normal coordinates around some point $p \in M$ with empty cut locus) then the L^p -spectrum actually depends on p . More precisely, Sturm showed that in this case $\inf \operatorname{Re} \sigma(\Delta_{M,1}) = 0$ whereas $\inf \sigma(\Delta_{M,2}) > 0$. An example where this happens is $M = \mathbb{H}^n$, the n -dimensional hyperbolic space.

In the latter case and for more general hyperbolic manifolds of the form $M = \Gamma \backslash \mathbb{H}^n$ where Γ denotes a geometrically finite discrete subgroup of the isometry group of \mathbb{H}^n such that either M has finite volume or M is cusp free, the L^p -spectrum was completely determined by Davies, Simon, and Taylor in [8]. They proved that $\sigma(\Delta_{M,p})$ coincides with the union of a parabolic region P_p and a (possibly empty) finite subset $\{\lambda_0, \dots, \lambda_m\}$ of $\mathbb{R}_{\geq 0}$ that consists of eigenvalues for $\Delta_{M,p}$. Note, that we have $P_2 = [\frac{(n-1)^2}{4}, \infty)$.

Taylor generalized this result in [24] to symmetric spaces X of non-compact type, i.e. he proved that the L^p -spectrum of X coincides with a certain parabolic region P_p (now defined in terms of X) that degenerates in the case $p = 2$ to the interval $[|\rho|^2, \infty)$, where a definition of ρ can be found in Section 2.2. He also showed that the methods from [8] can be used in order to prove the following:

Proposition 1.1 (cf. Proposition 3.3 in [24]). *Let X denote a symmetric space of non-compact type and $M = \Gamma \backslash X$ a locally symmetric space with finite volume. If*

$$\sigma(\Delta_{M,2}) \subset \{\lambda_0, \dots, \lambda_m\} \cup [|\rho|^2, \infty), \quad (1.1)$$

where $\lambda_j \in [0, |\rho|^2)$ are eigenvalues of finite multiplicity, then we have for $p \in [1, \infty)$:

$$\sigma(\Delta_{M,p}) \subset \{\lambda_0, \dots, \lambda_m\} \cup P_p.$$

However, for non-compact locally symmetric spaces $\Gamma \backslash X$ with finite volume the assumption (1.1) is in general not fulfilled: If X is a symmetric space of non-compact type and $\Gamma \subset \operatorname{Isom}^0(X)$ an arithmetic subgroup such that the quotient $M = \Gamma \backslash X$ is a complete, non-compact locally symmetric space, the continuous L^2 -spectrum of M contains the interval $[|\rho|^2, \infty)$ but is in general strictly larger.

Another upper bound for the L^p -spectrum $\sigma(\Delta_{M,p})$ is the sector

$$\left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \arctan \frac{|p-2|}{2\sqrt{p-1}} \right\} \cup \{0\}$$

which is indicated in Figure 1. This actually holds in a much more general setting, i.e. for generators of so-called submarkovian semigroups (cf. Section 2.1).

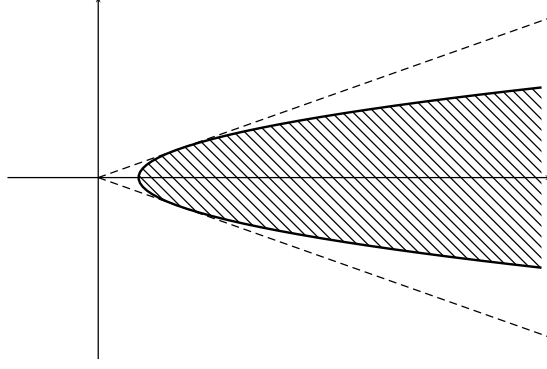


Figure 1: The parabolic region P_p if $p = 3$.

We are going to prove in Section 3 that a certain parabolic region (in general different from the one in Proposition 1.1) is contained in the L^p -spectrum $\sigma(\Delta_{M,p})$ of a locally symmetric space $M = \Gamma \backslash X$ with the properties mentioned in the beginning. In the case where X is a rank one symmetric space it happens that our parabolic region and the one in Taylor's result coincide. Therefore, we are able to determine explicitly the L^p -spectrum in the latter case.

In Section 4 we briefly explain, how the results from Section 3 can be generalized to manifolds with cusps of rank one. For these manifolds, every cusp defines a parabolic region that is contained in the L^p -spectrum. In contrast to the class of locally symmetric spaces however, these parabolic regions need not coincide. This is due to the fact that the volume growth in different cusps may be different in manifolds with cusps of rank one whereas this can not happen for locally symmetric spaces as above. Consequently, the number of (different) parabolic regions in the L^p -spectrum $\sigma(\Delta_{M,p})$, $p \neq 2$, of a manifold with cusps of rank one seems to be a lower bound for the number of cusps of M . As in the case $p = 2$ the Laplace-Beltrami operator is self-adjoint, we obtain in this case only the trivial lower bound one. Therefore, it seems that more geometric information is encoded in the L^p -spectrum for some $p \neq 2$ than in the L^2 -spectrum. Note however, that nothing new can be expected for compact manifolds as the L^p -spectrum does not depend on p in this case.

For results concerning the L^p -spectrum of locally symmetric spaces with *infinite* volume see [26, 27].

2 Preliminaries

2.1 Heat semigroup on L^p -spaces

In this section M denotes an arbitrary complete Riemannian manifold. The Laplace-Beltrami operator $\Delta_M := -\text{div}(\text{grad})$ with domain $C_c^\infty(M)$ (the set of differentiable functions with compact support) is essentially self-adjoint and hence, its closure (also

denoted by Δ_M) is a self-adjoint operator on the Hilbert space $L^2(M)$. Since Δ_M is positive, $-\Delta_M$ generates a bounded analytic semigroup $e^{-t\Delta_M}$ on $L^2(M)$ which can be defined by the spectral theorem for unbounded self-adjoint operators. The semigroup $e^{-t\Delta_M}$ is a *submarkovian semigroup* (i.e., $e^{-t\Delta_M}$ is positive and a contraction on $L^\infty(M)$ for any $t \geq 0$) and we therefore have the following:

- (1) The semigroup $e^{-t\Delta_M}$ leaves the set $L^1(M) \cap L^\infty(M) \subset L^2(M)$ invariant and hence, $e^{-t\Delta_M}|_{L^1 \cap L^\infty}$ may be extended to a positive contraction semigroup $T_p(t)$ on $L^p(M)$ for any $p \in [1, \infty]$. These semigroups are strongly continuous if $p \in [1, \infty)$ and *consistent* in the sense that $T_p(t)|_{L^p \cap L^q} = T_q(t)|_{L^p \cap L^q}$.
- (2) Furthermore, if $p \in (1, \infty)$, the semigroup $T_p(t)$ is a bounded analytic semigroup with angle of analyticity $\theta_p \geq \frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}}$.

For a proof of (1) we refer to [7, Theorem 1.4.1]. For (2) see [19]. In general, the semigroup $T_1(t)$ needs not be analytic. However, if M has bounded geometry $T_1(t)$ is analytic in *some* sector (cf. [25, 6]).

In the following, we denote by $-\Delta_{M,p}$ the generator of $T_p(t)$ (note, that $\Delta_M = \Delta_{M,2}$) and by $\sigma(\Delta_{M,p})$ the spectrum of $\Delta_{M,p}$. Furthermore, we will write $e^{-t\Delta_{M,p}}$ for the semigroup $T_p(t)$. Because of (2) from above, the L^p -spectrum $\sigma(\Delta_{M,p})$ has to be contained in the sector

$$\left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \frac{\pi}{2} - \theta_p \right\} \cup \{0\} \subset \left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \arctan \frac{|p-2|}{2\sqrt{p-1}} \right\} \cup \{0\}.$$

If we identify as usual the dual space of $L^p(M)$, $1 \leq p < \infty$, with $L^{p'}(M)$, $\frac{1}{p} + \frac{1}{p'} = 1$, the dual operator of $\Delta_{M,p}$ equals $\Delta_{M,p'}$ and therefore we always have $\sigma(\Delta_{M,p}) = \sigma(\Delta_{M,p'})$.

2.2 Symmetric spaces

Let X denote always a symmetric space of non-compact type. Then $G := \text{Isom}^0(X)$ is a non-compact, semi-simple Lie group with trivial center that acts transitively on X and $X = G/K$, where $K \subset G$ is a maximal compact subgroup of G . We denote the respective Lie algebras by \mathfrak{g} and \mathfrak{k} . Given a corresponding Cartan involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ we obtain the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of \mathfrak{g} into the eigenspaces of θ . The subspace \mathfrak{p} of \mathfrak{g} can be identified with the tangent space $T_{eK}X$. We assume, that the Riemannian metric $\langle \cdot, \cdot \rangle$ of X in $\mathfrak{p} \cong T_{eK}X$ coincides with the restriction of the Killing form $B(Y, Z) := \text{tr}(\text{ad}Y \circ \text{ad}Z)$, $Y, Z \in \mathfrak{g}$, to \mathfrak{p} .

For any maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ we refer to $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ as the set of restricted roots for the pair $(\mathfrak{g}, \mathfrak{a})$, i.e. Σ contains all $\alpha \in \mathfrak{a}^* \setminus \{0\}$ such that

$$\mathfrak{h}_\alpha := \{Y \in \mathfrak{g} : \text{ad}(H)(Y) = \alpha(H)Y \text{ for all } H \in \mathfrak{a}\} \neq \{0\}.$$

These subspaces $\mathfrak{h}_\alpha \neq \{0\}$ are called root spaces.

Once a positive Weyl chamber \mathfrak{a}^+ in \mathfrak{a} is chosen, we denote by Σ^+ the subset of positive roots and by $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{h}_\alpha) \alpha$ half the sum of the positive roots (counted according to their multiplicity).

2.2.1 Arithmetic groups and \mathbb{Q} -rank

Since $G = \text{Isom}^0(X)$ is a non-compact, semi-simple Lie group with trivial center, we can find a connected, semi-simple algebraic group $\mathbf{G} \subset GL(n, \mathbb{C})$ defined over \mathbb{Q} such that the groups G and $\mathbf{G}(\mathbb{R})^0$ are isomorphic as Lie groups (cf. [9, Proposition 1.14.6]).

Let us denote by $\mathbf{T}_{\mathbb{K}} \subset \mathbf{G}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$) a maximal \mathbb{K} -split algebraic torus in \mathbf{G} . Remember that we call a closed subgroup \mathbf{T} of \mathbf{G} a *torus* if \mathbf{T} is diagonalizable over \mathbb{C} , or equivalently if \mathbf{T} is abelian and every element of \mathbf{T} is semi-simple. Such a torus \mathbf{T} is called \mathbb{R} -split if \mathbf{T} is diagonalizable over \mathbb{R} and \mathbb{Q} -split if \mathbf{T} is defined over \mathbb{Q} and diagonalizable over \mathbb{Q} .

All maximal \mathbb{K} -split tori in \mathbf{G} are conjugate under $\mathbf{G}(\mathbb{K})$, and we call their common dimension \mathbb{K} -rank of \mathbf{G} . It turns out that the \mathbb{R} -rank of \mathbf{G} coincides with the rank of the symmetric space $X = G/K$, i.e. the dimension of a maximal flat subspace in X .

Since we are only interested in *non-uniform* lattices $\Gamma \subset G$, we may define arithmetic lattices in the following way (cf. [29, Corollary 6.1.10] and its proof):

Definition 2.1. *A non-uniform lattice $\Gamma \subset G$ in a connected semi-simple Lie group G with trivial center and no compact factors is called arithmetic if there are*

- (i) *a semi-simple algebraic group $\mathbf{G} \subset GL(n, \mathbb{C})$ defined over \mathbb{Q} and*
- (ii) *an isomorphism*

$$\varphi : \mathbf{G}(\mathbb{R})^0 \rightarrow G$$

such that $\varphi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0)$ and Γ are commensurable, i.e. $\varphi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0) \cap \Gamma$ has finite index in both $\varphi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0)$ and Γ .

For the general definition of arithmetic lattices see [29, Definition 6.1.1].

A well-known and fundamental result due to Margulis ensures that this is usually the only way to obtain a lattice. More precisely, every irreducible lattice $\Gamma \subset G$ in a connected, semi-simple Lie group G with trivial center, no compact factors and \mathbb{R} -rank(G) ≥ 2 is arithmetic ([20, 29]).

Further results due to Corlette (cf. [5]) and Gromov & Schoen (cf. [12]) extended this result to all connected semi-simple Lie groups with trivial center except $SO(1, n)$ and $SU(1, n)$. In $SO(1, n)$ (for all $n \in \mathbb{N}$) and in $SU(1, n)$ (for $n = 2, 3$) actually non-arithmetic lattices are known to exist (see e.g. [11, 20]).

Definition 2.2. (\mathbb{Q} -rank of an arithmetic lattice). *Suppose $\Gamma \subset G$ is an arithmetic lattice in a connected semi-simple Lie group G with trivial center and no compact factors. Then \mathbb{Q} -rank(Γ) is by definition the \mathbb{Q} -rank of \mathbf{G} , where \mathbf{G} is an algebraic group as in Definition 2.1.*

The theory of algebraic groups shows that the definition of the \mathbb{Q} -rank of an arithmetic lattice does not depend on the choice of the algebraic group \mathbf{G} in Definition 2.1. A proof of this fact can be found in [28, Corollary 9.12].

We already mentioned a geometric interpretation of the \mathbb{R} -rank: The \mathbb{R} -rank of \mathbf{G} as above coincides with the rank of the corresponding symmetric space $X = G/K$. For the \mathbb{Q} -rank of an arithmetic lattice Γ that acts freely on X there is also a geometric interpretation in terms of the large scale geometry of the corresponding locally symmetric space $\Gamma \backslash X$:

Let us fix an arbitrary point $p \in M = \Gamma \backslash X$. The *tangent cone at infinity* of M is the (pointed) Gromov-Hausdorff limit of the sequence $(M, p, \frac{1}{n}d_M)$ of pointed metric spaces. Heuristically speaking, this means that we are looking at the locally symmetric space M from farther and farther away. The precise definition can be found in [22, Chapter 10]. We have the following geometric interpretation of \mathbb{Q} -rank(Γ). For a proof see [13, 18] or [28].

Theorem 2.3. *Let $X = G/K$ denote a symmetric space of non-compact type and $\Gamma \subset G$ an arithmetic lattice that acts freely on X . Then, the tangent cone at infinity of $\Gamma \backslash X$ is isometric to a Euclidean cone over a finite simplicial complex whose dimension is \mathbb{Q} -rank(Γ).*

An immediate consequence of this theorem is that \mathbb{Q} -rank(Γ) = 0 if and only if the locally symmetric space $\Gamma \backslash X$ is compact.

2.2.2 Siegel sets and reduction theory

Let us denote in this subsection by \mathbf{G} again a connected, semi-simple algebraic group defined over \mathbb{Q} with trivial center and by $X = G/K$ the corresponding symmetric space of non-compact type with $G = \mathbf{G}^0(\mathbb{R})$. Our main references in this subsection are [1, 4, 16].

Langlands decomposition of rational parabolic subgroups.

Definition 2.4. *A closed subgroup $\mathbf{P} \subset \mathbf{G}$ defined over \mathbb{Q} is called rational parabolic subgroup if \mathbf{P} contains a maximal, connected solvable subgroup of \mathbf{G} . (These subgroups are called Borel subgroups of \mathbf{G} .)*

For any rational parabolic subgroup \mathbf{P} of \mathbf{G} we denote by $\mathbf{N}_{\mathbf{P}}$ the unipotent radical of \mathbf{P} , i.e. the largest unipotent normal subgroup of \mathbf{P} and by $N_{\mathbf{P}} := \mathbf{N}_{\mathbf{P}}(\mathbb{R})$ the real points of $\mathbf{N}_{\mathbf{P}}$. The *Levi quotient* $\mathbf{L}_{\mathbf{P}} := \mathbf{P}/\mathbf{N}_{\mathbf{P}}$ is reductive and both $\mathbf{N}_{\mathbf{P}}$ and $\mathbf{L}_{\mathbf{P}}$ are defined over \mathbb{Q} . If we denote by $\mathbf{S}_{\mathbf{P}}$ the maximal \mathbb{Q} -split torus in the center of $\mathbf{L}_{\mathbf{P}}$ and by $A_{\mathbf{P}} := \mathbf{S}_{\mathbf{P}}(\mathbb{R})^0$ the connected component of $\mathbf{S}_{\mathbf{P}}(\mathbb{R})$ containing the identity, we obtain the decomposition of $\mathbf{L}_{\mathbf{P}}(\mathbb{R})$ into $A_{\mathbf{P}}$ and the real points $M_{\mathbf{P}}$ of a reductive algebraic group $\mathbf{M}_{\mathbf{P}}$ defined over \mathbb{Q} :

$$\mathbf{L}_{\mathbf{P}}(\mathbb{R}) = A_{\mathbf{P}} M_{\mathbf{P}} \cong A_{\mathbf{P}} \times M_{\mathbf{P}}.$$

After fixing a certain basepoint $x_0 \in X$, we can lift the groups $\mathbf{L}_{\mathbf{P}}$, $\mathbf{S}_{\mathbf{P}}$ and $\mathbf{M}_{\mathbf{P}}$ into \mathbf{P} such that their images $\mathbf{L}_{\mathbf{P},x_0}$, $\mathbf{S}_{\mathbf{P},x_0}$ and $\mathbf{M}_{\mathbf{P},x_0}$ are algebraic groups defined over \mathbb{Q} (this is in general not true for every choice of a basepoint x_0) and give rise to the *rational Langlands decomposition* of $P := \mathbf{P}(\mathbb{R})$:

$$P \cong N_{\mathbf{P}} \times A_{\mathbf{P},x_0} \times M_{\mathbf{P},x_0}.$$

More precisely, this means that the map

$$P \rightarrow N_{\mathbf{P}} \times A_{\mathbf{P},x_0} \times M_{\mathbf{P},x_0}, \quad g \mapsto (n(g), a(g), m(g))$$

is a real analytic diffeomorphism.

Denoting by $X_{\mathbf{P},x_0}$ the *boundary symmetric space*

$$X_{\mathbf{P},x_0} := M_{\mathbf{P},x_0}/K \cap M_{\mathbf{P},x_0}$$

we obtain, since the subgroup P acts transitively on the symmetric space $X = G/K$ (we actually have $G = PK$), the following *rational horocyclic decomposition* of X :

$$X \cong N_{\mathbf{P}} \times A_{\mathbf{P},x_0} \times X_{\mathbf{P},x_0}.$$

More precisely, if we denote by $\tau : M_{\mathbf{P},x_0} \rightarrow X_{\mathbf{P},x_0}$ the canonical projection, we have an analytic diffeomorphism

$$\mu : N_{\mathbf{P}} \times A_{\mathbf{P},x_0} \times X_{\mathbf{P},x_0} \rightarrow X, \quad (n, a, \tau(m)) \mapsto nam \cdot x_0. \quad (2.1)$$

Note, that the boundary symmetric space $X_{\mathbf{P},x_0}$ is a Riemannian product of a symmetric space of non-compact type by a Euclidean space.

For minimal rational parabolic subgroups, i.e. Borel subgroups \mathbf{P} , we have

$$\dim A_{\mathbf{P},x_0} = \mathbb{Q}\text{-rank}(\mathbf{G}).$$

In the following we omit the reference to the chosen basepoint x_0 in the subscripts.

\mathbb{Q} -Roots. Let us fix some *minimal* rational parabolic subgroup \mathbf{P} of \mathbf{G} . We denote in the following by \mathfrak{g} , $\mathfrak{a}_{\mathbf{P}}$, and $\mathfrak{n}_{\mathbf{P}}$ the Lie algebras of the (real) Lie groups G , $A_{\mathbf{P}}$, and $N_{\mathbf{P}}$ defined above. Associated with the pair $(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ there is – similar to Section 2.2 – a system $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ of so-called \mathbb{Q} -roots. If we define for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ the *root spaces*

$$\mathfrak{g}_{\alpha} := \{Z \in \mathfrak{g} : \text{ad}(H)(Z) = \alpha(H)(Z) \text{ for all } H \in \mathfrak{a}_{\mathbf{P}}\},$$

we have the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} \mathfrak{g}_{\alpha},$$

where \mathfrak{g}_0 is the Lie algebra of $Z(\mathbf{S}_{\mathbf{P}}(\mathbb{R}))$, the center of $\mathbf{S}_{\mathbf{P}}(\mathbb{R})$. Furthermore, the minimal rational parabolic subgroup \mathbf{P} defines an ordering of $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ such that

$$\mathfrak{n}_{\mathbf{P}} = \bigoplus_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} \mathfrak{g}_{\alpha}.$$

The root spaces $\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}$ to distinct positive roots $\alpha, \beta \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ are orthogonal with respect to the Killing form:

$$B(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}) = \{0\}.$$

In analogy to Section 2.2 we define

$$\rho_{\mathbf{P}} := \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} (\dim \mathfrak{g}_{\alpha}) \alpha.$$

Furthermore, we denote by $\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ the set of simple positive roots. Recall, that we call a positive root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ simple if $\frac{1}{2}\alpha$ is not a root.

Remark 2.5. The elements of $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ are differentials of characters of the maximal \mathbb{Q} -split torus $\mathbf{S}_{\mathbf{P}}$. For convenience, we identify the \mathbb{Q} -roots with characters. If restricted to $A_{\mathbf{P}}$ we denote therefore the values of these characters by $\alpha(a)$, ($a \in A_{\mathbf{P}}, \alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$) which is defined by

$$\alpha(a) := \exp \alpha(\log a).$$

Siegel sets. Since we will consider in the succeeding section only (non-uniform) arithmetic lattices Γ with \mathbb{Q} -rank(Γ) = 1, we restrict ourselves from now on to the case

$$\mathbb{Q}\text{-rank}(\mathbf{G}) = 1.$$

For these groups we summarize several facts in the next lemma.

Lemma 2.6. *Assume \mathbb{Q} -rank(\mathbf{G}) = 1. Then the following holds:*

- (1) *For any proper rational parabolic subgroup \mathbf{P} of \mathbf{G} , we have $\dim A_{\mathbf{P}} = 1$.*
- (2) *All proper rational parabolic subgroups are minimal.*
- (3) *The set $\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ of simple positive \mathbb{Q} -roots contains only a single element:*

$$\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}}) = \{\alpha\}.$$

For any rational parabolic subgroup \mathbf{P} of \mathbf{G} and any $t > 1$, we define

$$A_{\mathbf{P},t} := \{a \in A_{\mathbf{P}} : \alpha(a) > t\},$$

where α denotes the unique root in $\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$.

If we choose $a_0 \in A_{\mathbf{P}}$ with the property $\alpha(a_0) = t$, the set $A_{\mathbf{P},t}$ is just a shift of the positive Weyl chamber $A_{\mathbf{P},1}$ by a_0 :

$$A_{\mathbf{P},t} = A_{\mathbf{P},1} a_0.$$

Before we define *Siegel sets*, we recall the rational horocyclic decomposition of the symmetric space $X = G/K$:

$$X \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}.$$

Definition 2.7. Let \mathbf{P} denote a rational parabolic subgroup of the algebraic group \mathbf{G} with $\mathbb{Q}\text{-rank}(\mathbf{G}) = 1$. For any bounded set $\omega \subset N_{\mathbf{P}} \times X_{\mathbf{P}}$ and any $t > 1$, the set

$$\mathcal{S}_{\mathbf{P},\omega,t} := \omega \times A_{\mathbf{P},t} \subset X$$

is called Siegel set.

Precise reduction theory. We fix an arithmetic lattice $\Gamma \subset G = \mathbf{G}(\mathbb{R})$ in the algebraic group \mathbf{G} with $\mathbb{Q}\text{-rank}(\mathbf{G}) = 1$. Recall, that by a well known result due to A. Borel and Harish-Chandra there are only finitely many Γ -conjugacy classes of minimal parabolic subgroups (see e.g. [1]). Using the Siegel sets defined above, we can state the *precise reduction theory* in the \mathbb{Q} -rank one case as follows:

Theorem 2.8. Let \mathbf{G} denote a semi-simple algebraic group defined over \mathbb{Q} with $\mathbb{Q}\text{-rank}(\mathbf{G}) = 1$ and Γ an arithmetic lattice in G . We further denote by $\mathbf{P}_1, \dots, \mathbf{P}_k$ representatives of the Γ -conjugacy classes of all rational proper (i.e. minimal) parabolic subgroups of \mathbf{G} . Then there exist a bounded set $\Omega_0 \subset X$ and Siegel sets $\omega_j \times A_{\mathbf{P}_j,t_j}$ ($j = 1, \dots, k$) such that the following holds:

- (1) Under the canonical projection $\pi : X \rightarrow \Gamma \backslash X$ each Siegel set $\omega_j \times A_{\mathbf{P}_j,t_j}$ is mapped injectively into $\Gamma \backslash X$, $i = 1, \dots, k$.
- (2) The image of ω_j in $(\Gamma \cap P_j) \backslash N_{\mathbf{P}_j} \times X_{\mathbf{P}_j}$ is compact ($j = 1, \dots, k$).
- (3) The subset

$$\Omega_0 \cup \coprod_{j=1}^k \omega_j \times A_{\mathbf{P}_j,t_j}$$

is an open fundamental domain for Γ . In particular, $\Gamma \backslash X$ equals the closure of $\pi(\Omega_0) \cup \coprod_{j=1}^k \pi(\omega_j \times A_{\mathbf{P}_j,t_j})$.

Geometrically this means that the closure of each set $\pi(\omega_j \times A_{\mathbf{P}_j,t_j})$ corresponds to one cusp of the locally symmetric space $\Gamma \backslash X$ and the numbers t_j are chosen large enough such that these sets do not overlap. Then the interior of the bounded set $\pi(\Omega_0)$ is just the complement of the closure of $\coprod_{j=1}^k \pi(\omega_j \times A_{\mathbf{P}_j,t_j})$.

Since in the case $\mathbb{Q}\text{-rank}(\mathbf{G}) = 1$ all rational proper parabolic subgroups are minimal, these subgroups are conjugate under $\mathbf{G}(\mathbb{Q})$ (cf. [1, Theorem 11.4]). Therefore, the root systems $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}_j})$ with respect to the rational proper parabolic subgroups \mathbf{P}_j , $j = 1 \dots k$, are canonically isomorphic (cf. [1, 11.9]) and moreover, we can conclude $\|\rho_{\mathbf{P}_1}\| = \dots = \|\rho_{\mathbf{P}_k}\|$.

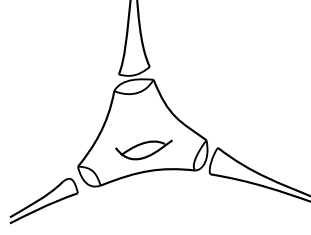


Figure 2: Disjoint Decomposition of a \mathbb{Q} -rank-1 Space.

2.2.3 Rational horocyclic coordinates

For all $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ we define on $\mathfrak{n}_{\mathbf{P}} = \bigoplus_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} \mathfrak{g}_{\alpha}$ a left invariant bilinear form h_{α} by

$$h_{\alpha} := \begin{cases} \langle \cdot, \cdot \rangle, & \text{on } \mathfrak{g}_{\alpha} \\ 0, & \text{else,} \end{cases}$$

where $\langle Y, Z \rangle := -B(Y, \theta Z)$ denotes the usual $\text{Ad}(K)$ -invariant bilinear form on \mathfrak{g} induced from the Killing form B . We then have (cf. [2, Proposition 1.6] or [3, Proposition 4.3]):

Proposition 2.9. (a) *For any $x = (n, \tau(m), a) \in X \cong N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ the tangent spaces at x to the submanifolds $\{n\} \times X_{\mathbf{P}} \times \{a\}$, $\{n\} \times \{\tau(m)\} \times A_{\mathbf{P}}$, and $N_{\mathbf{P}} \times \{\tau(m)\} \times \{a\}$ are mutually orthogonal.*

(b) *The pullback μ^*g of the metric g on X to $N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ is given by*

$$ds_{(n, \tau(m), a)}^2 = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} e^{-2\alpha(\log a)} h_{\alpha} \oplus d(\tau(m))^2 \oplus da^2.$$

If we choose orthonormal bases $\{N_1, \dots, N_r\}$ of $\mathfrak{n}_{\mathbf{P}}$, $\{Y_1, \dots, Y_l\}$ of some tangent space $T_{\tau(m)}X_{\mathbf{P}}$ and $H \in \mathfrak{a}_{\mathbf{P}}^+$ with $\|H\| = 1$, we obtain *rational horocyclic coordinates*

$$\begin{aligned} \varphi : N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}} &\rightarrow \mathbb{R}^r \times \mathbb{R}^l \times \mathbb{R}, \\ \left(\exp\left(\sum_{j=1}^r x_j N_j\right), \exp\left(\sum_{j=1}^l x_{j+r} Y_j\right), \exp(yH) \right) &\mapsto (x_1, \dots, x_{r+l}, y). \end{aligned}$$

In the following, we will abbreviate (x_1, \dots, x_{r+l}, y) as (x, y) . The representation of the metric ds^2 with respect to these coordinates is given by the matrix

$$(g_{ij})_{i,j}(n, \tau(m), a) = \begin{pmatrix} \boxed{\begin{matrix} \frac{1}{2}e^{-2\alpha_1(\log a)} & & & & 0 \\ & \ddots & & & \\ & & \frac{1}{2}e^{-2\alpha_r(\log a)} & & \\ & & & \boxed{h_{km}} & \\ 0 & & & & \boxed{1} \end{matrix}} \end{pmatrix}$$

where the positive roots $\alpha_i \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ appear according to their multiplicity and the $(l \times l)$ -submatrix $(h_{km})_{k,m=1}^l$ represents the metric $d(\tau(m))^2$ on the boundary symmetric space $X_{\mathbf{P}}$.

Corollary 2.10. *The volume form of $N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ with respect to rational horocyclic coordinates is given by*

$$\begin{aligned} \sqrt{\det(g_{ij})(n, \tau(m), a)} dx dy &= \left(\frac{1}{2}\right)^{r/2} \sqrt{\det(h_{km}(\tau(m)))} e^{-2\rho_{\mathbf{P}}(\log a)} dx dy \\ &= \left(\frac{1}{2}\right)^{r/2} \sqrt{\det(h_{km}(\tau(m)))} e^{-2\|\rho_{\mathbf{P}}\|y} dx dy, \end{aligned}$$

where $\log a = yH$.

A straightforward calculation yields

Corollary 2.11. *The Laplacian Δ on $N_{\mathbf{P}} \times X_{\mathbf{P}} \times A_{\mathbf{P}}$ in rational horocyclic coordinates is*

$$\Delta = -2 \sum_{j=1}^r e^{2\alpha_j} \frac{\partial^2}{\partial x_j^2} + \Delta_{X_{\mathbf{P}}} - \frac{\partial^2}{\partial y^2} + 2\|\rho_{\mathbf{P}}\| \frac{\partial}{\partial y}, \quad (2.2)$$

where $\Delta_{X_{\mathbf{P}}}$ denotes the Laplacian on the boundary symmetric space $X_{\mathbf{P}}$ and $e^{2\alpha_j}$ is short hand for the function $(x, y) \mapsto e^{2y\alpha_j(H)}$.

3 L^p -Spectrum

In this section $X = G/K$ denotes again a symmetric space of non-compact type whose metric coincides on $T_e K(G/K) \cong \mathfrak{p}$ with the Killing form of the Lie algebra \mathfrak{g} of G . Furthermore, $\Gamma \subset G$ is an arithmetic (non-uniform) lattice with \mathbb{Q} -rank(Γ) = 1. We also assume that Γ is torsion-free.

The corresponding locally symmetric space $M = \Gamma \backslash X$ has finitely many cusps and each cusp corresponds to a Γ -conjugacy class of a minimal rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Let $\mathbf{P}_1, \dots, \mathbf{P}_k$ denote representatives of the Γ -conjugacy classes. Since these subgroups are conjugate under $\mathbf{G}(\mathbb{Q})$ and the respective root systems are isomorphic (cf. Section 2.2.2), we consider in the following only the rational parabolic subgroup $\mathbf{P} := \mathbf{P}_1$. We denote by $\rho_{\mathbf{P}}$ as in the preceding section half the sum of the positive roots (counted according to their multiplicity) with respect to the pair $(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$.

We define for any $p \in [1, \infty)$ the parabolic region

$$P_p := \left\{ z = x + iy \in \mathbb{C} : x \geq \frac{4\|\rho_{\mathbf{P}}\|^2}{p} \left(1 - \frac{1}{p}\right) + \frac{y^2}{4\|\rho_{\mathbf{P}}\|^2(1 - \frac{2}{p})^2} \right\}$$

if $p \neq 2$ and $P_2 := [\|\rho_{\mathbf{P}}\|^2, \infty)$.

Note, that the boundary ∂P_p of P_p is parametrized by the curve

$$\begin{aligned} \mathbb{R} \rightarrow \mathbb{C}, \quad s \mapsto & \frac{4\|\rho_{\mathbf{P}}\|^2}{p} \left(1 - \frac{1}{p}\right) + s^2 + 2i\|\rho_{\mathbf{P}}\|s \left(1 - \frac{2}{p}\right) \\ & = \left(\frac{2\|\rho_{\mathbf{P}}\|}{p} + is\right) \left(2\|\rho_{\mathbf{P}}\| - \frac{2\|\rho_{\mathbf{P}}\|}{p} - is\right) \end{aligned}$$

and that this parabolic region coincides with the one in Proposition 1.1 if and only if $\|\rho_{\mathbf{P}}\| = \|\rho\|$.

Our main result in this chapter reads as follows:

Theorem 3.1. *Let $X = G/K$ denote a symmetric space of non-compact type and $\Gamma \subset G$ an arithmetic lattice with \mathbb{Q} -rank(Γ) = 1 that acts freely on X . If we denote by $M := \Gamma \backslash X$ the corresponding locally symmetric space, the parabolic region P_p is contained in the spectrum of $\Delta_{M,p}$, $p \in (1, \infty)$:*

$$P_p \subset \sigma(\Delta_{M,p}).$$

Lemma 3.2. *Let M denote a Riemannian manifold with finite volume. For $1 \leq p \leq q < \infty$, we have*

$$e^{-t\Delta_{M,q}} \Delta_{M,q} \subset \Delta_{M,p} e^{-t\Delta_{M,q}}.$$

Proof. Since the volume of M is finite, it follows by Hölder's inequality

$$L^q(M) \hookrightarrow L^p(M),$$

i.e. $L^q(M)$ is continuously embedded in $L^p(M)$. Therefore, we obtain the boundedness of the operators

$$e^{-t\Delta_{M,q}} : L^q(M) \rightarrow L^p(M). \quad (3.1)$$

To prove the lemma, we choose an $f \in \text{dom}(\Delta_{M,q}) = \text{dom}(e^{-t\Delta_{M,q}} \Delta_{M,q})$. Because of $e^{-t\Delta_{M,q}} f \in L^p(M) \cap \text{dom}(\Delta_{M,q})$ and the consistency of the semigroups $e^{-t\Delta_{M,p}}$, $p \in [1, \infty)$, we have $e^{-s\Delta_{M,p}} e^{-t\Delta_{M,q}} f = e^{-(t+s)\Delta_{M,q}} f$ and obtain by using (3.1):

$$\begin{aligned} & \left\| \frac{1}{s} (e^{-s\Delta_{M,p}} e^{-t\Delta_{M,q}} f - e^{-t\Delta_{M,q}} f) - e^{-t\Delta_{M,q}} \Delta_{M,q} f \right\|_{L^p} \leq \\ & C \left\| \frac{1}{s} (e^{-s\Delta_{M,q}} f - f) - \Delta_{M,q} f \right\|_{L^q} \rightarrow 0 \quad (s \rightarrow 0^+). \end{aligned}$$

Thus, the function $e^{-t\Delta_{M,q}} f$ is contained in the domain of $\Delta_{M,p}$ and we also have the equality

$$e^{-t\Delta_{M,q}} \Delta_{M,q} f = \Delta_{M,p} e^{-t\Delta_{M,q}} f.$$

□

The following proposition follows from the preceding lemma as in [15, Proposition 3.1] or [14, Proposition 2.1]. For the sake of completeness we work out the details.

Proposition 3.3. *Let M denote a Riemannian manifold with finite volume. For $2 \leq p \leq q < \infty$, we have the inclusion*

$$\sigma(\Delta_{M,p}) \subset \sigma(\Delta_{M,q}).$$

Proof. The statement of the proposition is obviously equivalent to the reverse inclusion for the respective resolvent sets:

$$\rho(\Delta_{M,q}) \subset \rho(\Delta_{M,p}).$$

We are going to show that for $\lambda \in \rho(\Delta_{M,q}) \cap \rho(\Delta_{M,p})$ the resolvents coincide on $L^q(M) \cap L^p(M)$. From Lemma 3.2 above, we conclude for these λ

$$\begin{aligned} (\lambda - \Delta_{M,p})^{-1} e^{-t\Delta_{M,q}} &= (\lambda - \Delta_{M,p})^{-1} e^{-t\Delta_{M,q}} (\lambda - \Delta_{M,q}) (\lambda - \Delta_{M,q})^{-1} \\ &= (\lambda - \Delta_{M,p})^{-1} (\lambda - \Delta_{M,p}) e^{-t\Delta_{M,q}} (\lambda - \Delta_{M,q})^{-1} \\ &= e^{-t\Delta_{M,q}} (\lambda - \Delta_{M,q})^{-1}, \end{aligned} \tag{3.2}$$

where the equality is meant between bounded operators from $L^q(M)$ to $L^p(M)$. If $t \rightarrow 0$, we obtain

$$(\lambda - \Delta_{M,p})^{-1}|_{L^q \cap L^p} = (\lambda - \Delta_{M,q})^{-1}|_{L^q \cap L^p}.$$

For $\frac{1}{q} + \frac{1}{q'} = 1$ (in particular, this implies $q' \leq p \leq q$) and $\lambda \in \rho(\Delta_{M,q}) = \rho(\Delta_{M,q'})$ we have by the preceding calculation

$$(\lambda - \Delta_{M,q'})^{-1}|_{L^q \cap L^{q'}} = (\lambda - \Delta_{M,q})^{-1}|_{L^q \cap L^{q'}}.$$

The Riesz-Thorin interpolation theorem implies that $(\lambda - \Delta_{M,q})^{-1}$ is bounded if considered as an operator R_λ on $L^p(M)$.

In the remainder of the proof we show that R_λ coincides with $(\lambda - \Delta_{M,p})^{-1}$ and hence $\rho(\Delta_{M,q}) \subset \rho(\Delta_{M,p})$. Notice, that (3.2) implies

$$(\lambda - \Delta_{M,p}) e^{-t\Delta_{M,q}} (\lambda - \Delta_{M,q})^{-1} f = e^{-t\Delta_{M,q}} f,$$

for all $f \in L^p(M) \cap L^q(M)$. Since $\Delta_{M,p}$ is a closed operator, we obtain for $t \rightarrow 0$ the limit

$$(\lambda - \Delta_{M,p}) R_\lambda f = f.$$

As $L^q(M) \cap L^p(M)$ is dense in $L^p(M)$ and $\Delta_{M,p}$ is closed, it follows $(\lambda - \Delta_{M,p}) R_\lambda f = f$ for all $f \in L^p(M)$. Therefore, $(\lambda - \Delta_{M,p})$ is onto. If we assume that $(\lambda - \Delta_{M,p})$ is not one-to-one, λ would be an eigenvalue of $\Delta_{M,p}$. Assume $f \neq 0$ is an eigenfunction of $\Delta_{M,p}$ for the eigenvalue λ . Then it follows from Lemma 3.2:

$$\lambda e^{-t\Delta_{M,p}} f = \Delta_{M,q'} e^{-t\Delta_{M,p}} f.$$

Since $e^{-t\Delta_{M,p}}$ is strongly continuous there is a $t_0 > 0$ such that $e^{-t_0\Delta_{M,p}} f \neq 0$ and $e^{-t_0\Delta_{M,p}} f$ is therefore an eigenfunction of $\Delta_{M,q'}$ for the eigenvalue λ . But this contradicts $\lambda \in \rho(\Delta_{M,q}) = \rho(\Delta_{M,q'})$. We finally obtain $R_\lambda = (\lambda - \Delta_{M,p})^{-1}$. \square

Proposition 3.4. *For $1 \leq p < \infty$ the boundary ∂P_p of the parabolic region P_p is contained in the approximate point spectrum of $\Delta_{M,p}$:*

$$\partial P_p \subset \sigma_{app}(\Delta_{M,p}).$$

Proof. In this proof we construct for any $z \in \partial P_p$ a sequence f_n of differentiable functions in $L^p(X)$ with support in a fundamental domain for Γ such that

$$\frac{\|\Delta_{X,p} f_n - z f_n\|_{L^p}}{\|f_n\|_{L^p}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Since such a sequence (f_n) descends to a sequence of differentiable functions in $L^p(M)$ this is enough to prove the proposition.

Recall that a fundamental domain for Γ is given by a subset of the form

$$\Omega_0 \cup \prod_{i=1}^k \omega_i \times A_{\mathbf{P}_i, t_i} \subset X$$

(cf. Theorem 2.8), and each Siegel set $\omega_i \times A_{\mathbf{P}_i, t_i}$ is mapped injectively into $\Gamma \backslash X$. Furthermore, the closure of $\pi(\omega_i \times A_{\mathbf{P}_i, t_i})$ fully covers an end of $\Gamma \backslash X$ (for any $i \in \{1, \dots, k\}$).

Now, we choose some

$$z = z(s) = \left(\frac{2\|\rho_{\mathbf{P}}\|}{p} + is \right) \left(2\|\rho_{\mathbf{P}}\| - \frac{2\|\rho_{\mathbf{P}}\|}{p} - is \right) \in \partial P_p.$$

Furthermore, we take the Siegel set $\omega \times A_{\mathbf{P}, t} := \omega_1 \times A_{\mathbf{P}_1, t_1}$ where $A_{\mathbf{P}, t} = \{a \in A_{\mathbf{P}} : \alpha(a) > t\}$, and define a sequence f_n of smooth functions with support in $\omega \times A_{\mathbf{P}, t}$ with respect to rational horocyclic coordinates by

$$f_n(x, y) := c_n(y) e^{\left(\frac{2}{p}\|\rho_{\mathbf{P}}\| + is\right)y},$$

where $c_n \in C_c^\infty\left(\left(\frac{\log t}{\|\alpha\|}, \infty\right)\right)$ is a so-far arbitrary sequence of differentiable functions with support in $\left(\frac{\log t}{\|\alpha\|}, \infty\right)$. Since ω is bounded, each f_n is clearly contained in $L^p(X)$. Furthermore, the condition $\text{supp}(c_n) \subset \left(\frac{\log t}{\|\alpha\|}, \infty\right)$ ensures that the supports of the sequence f_n are contained in the Siegel set $\omega \times A_{\mathbf{P}, t}$.

Using formula (2.2) for the Laplacian in rational horocyclic coordinates, we obtain after a straightforward calculation

$$\begin{aligned} \Delta_{X,p} f_n(x, y) - z f_n(x, y) &= \\ &= \left(-c_n''(y) + \left(2\|\rho_{\mathbf{P}}\| - 2\left(\frac{2}{p}\|\rho_{\mathbf{P}}\| + is\right) \right) c_n'(y) \right) e^{\left(\frac{2}{p}\|\rho_{\mathbf{P}}\| + is\right)y}, \end{aligned}$$

and therefore

$$\begin{aligned}
\|\Delta_{X,p}f_n - zf_n\|_{L^p}^p &= \int_{\omega \times A_{\mathbf{P},t}} |\Delta_{X,p}f_n - zf_n|^p d\text{vol}_X \\
&= \left(\frac{1}{2}\right)^{r/2} \int_{\omega \times A_{\mathbf{P},t}} |\Delta_{X,p}f_n(x, y) - zf_n(x, y)|^p \sqrt{\det(h_{km}(\tau(m)))} e^{-2\|\rho_{\mathbf{P}}\|y} dx dy \\
&= C \int_0^\infty \left| -c_n''(y) + \left(2\|\rho_{\mathbf{P}}\| - 2\left(\frac{2\|\rho_{\mathbf{P}}\|}{p} + is\right) \right) c_n'(y) \right|^p dy,
\end{aligned}$$

where $C := (\frac{1}{2})^{r/2} \int_{\omega} \sqrt{\det(h_{km}(\tau(m)))} dx < \infty$ because $\omega \subset N_{\mathbf{P}} \times X_{\mathbf{P}}$ is bounded. This yields after an application of the triangle inequality

$$\|\Delta_{X,p}f_n - zf_n\|_{L^p} \leq C_1 \left(\int_0^\infty |c_n''(y)|^p dy \right)^{1/p} + C_2 \left(\int_0^\infty |c_n'(y)|^p dy \right)^{1/p}.$$

By an analogous calculation we obtain

$$\|f_n\|_{L^p} = C_3 \left(\int_0^\infty |c_n(y)|^p dy \right)^{1/p}.$$

We choose a function $\psi \in C_c^\infty(\mathbb{R})$, not identically zero, with $\text{supp}(\psi) \subset (1, 2)$, a sequence $r_n > 0$ with $r_n \rightarrow \infty$ (if $n \rightarrow \infty$), and we eventually define

$$c_n(y) := \psi\left(\frac{y}{r_n}\right).$$

For large enough n , we have $\text{supp}(c_n) \subset (\frac{\log t}{\|\alpha\|}, \infty)$. An easy calculation gives

$$\begin{aligned}
\int_0^\infty |c_n(y)|^p dy &= r_n \int_1^2 |\psi(u)|^p du, \\
\int_0^\infty |c_n'(y)|^p dy &= r_n^{1-p} \int_1^2 |\psi'(u)|^p du, \\
\int_0^\infty |c_n''(y)|^p dy &= r_n^{1-2p} \int_1^2 |\psi''(u)|^p du.
\end{aligned}$$

In the end, this leads to the inequality

$$\frac{\|\Delta_{X,p}f_n - zf_n\|_p}{\|f_n\|_p} \leq \frac{C_4}{r_n} + \frac{C_5}{r_n^2} \longrightarrow 0 \quad (n \rightarrow \infty),$$

where $C_4, C_5 > 0$ denote positive constants, and the proof is complete. \square

Proof of Theorem 3.1. The inclusion

$$P_p \subset \sigma(\Delta_{M,p})$$

for $p \in [2, \infty)$ follows immediately from Proposition 3.3 and Proposition 3.4 by observing

$$P_p = \bigcup_{q \in [2, p]} \partial P_q.$$

The inclusion for all $p \in (1, \infty)$ follows by duality as $P_p = P_{p'}$ if $\frac{1}{p} + \frac{1}{p'} = 1$. \square

Up to now, we considered non-uniform arithmetic lattices $\Gamma \subset G$ with \mathbb{Q} -rank one. We made no assumption concerning the rank of the respective symmetric space $X = G/K$ of non-compact type. However, if $\text{rank}(X) = 1$, we are able to sharpen the result of Theorem 3.1 considerably. In the case $\mathbb{Q}\text{-rank}(\Gamma) = \text{rank}(X) = 1$, the one dimensional abelian subgroup $A_{\mathbf{P}}$ of G (with respect to some rational minimal parabolic subgroup) defines a maximal flat subspace, i.e. a geodesic, $A_{\mathbf{P}} \cdot x_0$ of X . Hence, the \mathbb{Q} -roots coincide with the roots defined in Section 2.2 and for any rational minimal parabolic subgroup \mathbf{P} we have in particular

$$\|\rho_{\mathbf{P}}\| = \|\rho\|.$$

Corollary 3.5. *Let $X = G/K$ denote a symmetric space of non-compact type with $\text{rank}(X) = 1$. Furthermore, $\Gamma \subset G$ denotes a non-uniform arithmetic lattice that acts freely on X and $M = \Gamma \backslash X$ the corresponding locally symmetric space. Then, we have for all $p \in (1, \infty)$ the equality*

$$\sigma(\Delta_{M,p}) = \{\lambda_0, \dots, \lambda_m\} \cup P_p,$$

where $0 = \lambda_0, \dots, \lambda_m \in [0, \|\rho\|^2)$ are eigenvalues of $\Delta_{M,2}$ with finite multiplicity.

Proof. Langlands' theory of Eisenstein series implies (see e.g. [17] or the surveys in [16] or [4])

$$\sigma(\Delta_{M,2}) = \{\lambda_0, \dots, \lambda_m\} \cup [\|\rho\|^2, \infty),$$

where $0 = \lambda_0, \dots, \lambda_m \in [0, \|\rho\|^2)$ are eigenvalues of $\Delta_{M,2}$ with finite multiplicity. Thus, we can apply Proposition 1.1 and obtain

$$\sigma(\Delta_{M,p}) \subset \{\lambda_0, \dots, \lambda_m\} \cup P_p.$$

As in the proof of [8, Lemma 6] one sees that the discrete part of the L^2 -spectrum $\{\lambda_0, \dots, \lambda_m\}$ is also contained in $\sigma(\Delta_{M,p})$ for any $p \in (1, \infty)$. Together with Theorem 3.1 and the remark above this concludes the proof. \square

As remarked in [24] one can prove as in [8] that every L^2 -eigenfunction of the Laplace-Beltrami operator $\Delta_{M,2}$ with respect to the eigenvalue λ_j , $j = 0, \dots, m$, lies in $L^p(M)$ if λ_j is *not* contained in P_p .

Remark 3.6. Because of the description of fundamental domains for *general* lattices in semi-simple Lie groups with \mathbb{R} -rank one (see [10]) it seems that the arithmeticity of Γ in Corollary 3.5 is not needed.

4 Manifolds with cusps of rank one

In this chapter we consider a class of Riemannian manifolds that is larger than the class of \mathbb{Q} -rank one locally symmetric spaces. This larger class consists of those manifolds which are isometric – after the removal of a compact set – to a disjoint union of rank one cusps. Manifolds with cusps of rank one were probably first introduced and studied by W. Müller (see e.g. [21]).

4.1 Definition

Recall, that we denoted by $\omega \times A_{\mathbf{P},t} \subset X$ Siegel sets of a symmetric space $X = G/K$ of non-compact type. The projection $\pi(\omega \times A_{\mathbf{P},t})$ of certain Siegel sets to a corresponding \mathbb{Q} -rank one locally symmetric space $\Gamma \backslash X$ is a cusp and every cusp of $\Gamma \backslash X$ is of this form (cf. Section 2.2.2).

Definition 4.1. *A Riemannian manifold is called cusp of rank one if it is isometric to a cusp $\pi(\omega \times A_{\mathbf{P},t})$ of a \mathbb{Q} -rank one locally symmetric space.*

Definition 4.2. *A complete Riemannian manifold M is called manifold with cusps of rank one if it has a decomposition*

$$M = M_0 \cup \bigcup_{j=1}^k M_j$$

such that the following holds:

- (i) M_0 is a compact manifold with boundary.
- (ii) The subsets M_j , $j \in \{0, \dots, k\}$, are pairwise disjoint.
- (iii) For each $j \in \{1, \dots, k\}$ there exists a cusp of rank one isometric to M_j .

Such manifolds certainly have finite volume as there is only a finite number of cusps possible and every cusp of rank one has finite volume.

From Theorem 2.8 it follows that any \mathbb{Q} -rank one locally symmetric space is a manifold with cusps of rank one. But since we can perturb the metric on the compact manifold M_0 without leaving the class of manifolds with cusps of rank one, not every such manifold is locally symmetric. Of course, they are locally symmetric on each cusp and we can say that they are locally symmetric near infinity.

4.2 L^p -Spectrum and Geometry

Precisely as in Proposition 3.4 one sees that we can find for every cusp M_j , $j \in \{1, \dots, k\}$ of a manifold $M = M_0 \cup \bigcup_{j=1}^k M_j$ with cusps of rank one a parabolic region $P_p^{(j)}$ such that the boundary $\partial P_p^{(j)}$ is contained in the approximate point spectrum of $\Delta_{M,p}$. Here, the parabolic regions are defined as the parabolic region in the preceding section, where the constant $\|\rho_{\mathbf{P}}\|$ is replaced by an analogous quantity, say $\|\rho_{\mathbf{P}_j}\|$, coming from the respective cusp M_j . That is to say, we have the following lemma:

Lemma 4.3. *Let M denote a manifold with cusps of rank one. Then we have for $p \in [1, \infty)$ and $j = 1, \dots, k$:*

$$\partial P_p^{(j)} \subset \sigma_{app}(\Delta_{M,p}).$$

Since the volume of a manifold with cusps of rank one is finite, we can apply Proposition 3.3 in order to prove (cf. the proof of Theorem 3.1) the following

Theorem 4.4. *Let $M = M_0 \cup \bigcup_{j=1}^k M_j$ denote a manifold with cusps of rank one. Then, for $p \in (1, \infty)$, every cusp M_j defines a parabolic region $P_p^{(j)}$ that is contained in the L^p -spectrum:*

$$\bigcup_{j=1}^k P_p^{(j)} \subset \sigma(\Delta_{M,p}).$$

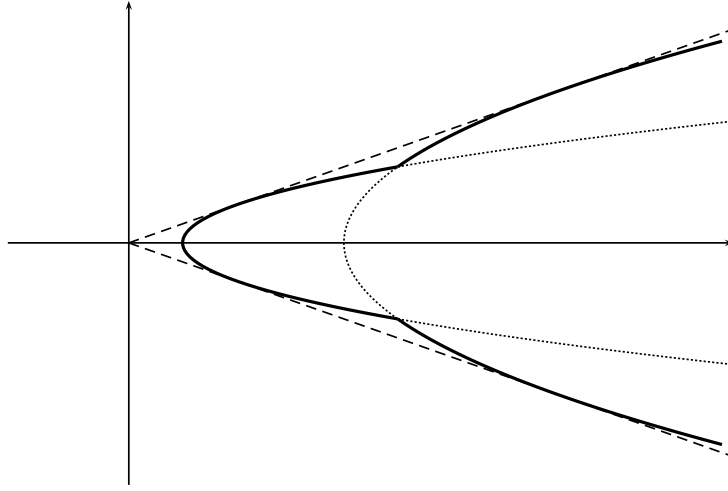


Figure 3: The union of two parabolic regions $P_p^{(1)}$ and $P_p^{(2)}$ if $p \neq 2$.

Of course, the compact submanifold M_0 contributes some discrete set to the L^p -spectrum, and 0 is always an eigenvalue as the volume of M is finite. It seems to be very likely that besides some discrete spectrum the union of the parabolic regions in Theorem 4.4 is already the complete spectrum. But at present, I do not know how to prove this result. The methods used in [8] or [24] to prove a similar result need either that the manifold is homogeneous or that the injectivity radius is bounded from below, and it is not clear how one could adapt the methods therein to our case.

Nevertheless, given the L^p -spectrum for some $p \neq 2$, we have the following geometric consequences:

Corollary 4.5. *Let $M = M_0 \cup \bigcup_{j=1}^k M_j$ denote a manifold with cusps of rank one such that*

$$\sigma(\Delta_{M,p}) = \{\lambda_0, \dots, \lambda_r\} \cup P_p,$$

for some $p \neq 2$ and some parabolic region P_p . Then every cusp M_j is of the form $\pi(\omega_j \times A_{\mathbf{P}_j, t_j})$ with volume form

$$\left(\frac{1}{2}\right)^{r_j/2} e^{-2yc} dx dy,$$

where c is a positive constant.

Proof. Since all parabolic regions $P_p^{(j)}$ induced by the cusps M_j coincide, the quantities $\|\rho_{\mathbf{P}_j}\|$ coincide. Therefore, we can take $c := \|\rho_{\mathbf{P}_1}\|$. \square

This result generalizes to the case where the continuous spectrum consists of a finite number of parabolic regions in an obvious manner.

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